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Regularization with Differential Operators. I. General Theory

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The method of regularization is used to obtain least squares solutions of the linear equation $Kx = y$, where K is a bounded linear operator from one Hilbert space into another and the regularizing operator L is a closed densely defined linear operator. Existence, uniqueness, and convergence analyses are developed. An application is given to the special case when K is a first kind integral operator and L is an nth order differential operator in the Hilbert space $L^2[a, b]$.

1. INTRODUCTION

In this paper we study the method of regularization in a general Hilbert space setting, approaching it from two different perspectives: first, by means of the Euler or regularized normal equation, and second, by treating it as a least squares process in an associated product space setting. These ideas are then applied to first kind integral equations under regularization with an arbitrary linear differential operator. Direct application of this theory to the numerical solution of first kind integral equations will be presented in a subsequent paper [7].

To briefly describe regularization, let X and Y be Hilbert spaces, let K be a bounded everywhere defined linear operator from X into Y, and let y be an arbitrary element of Y. In solving the linear equation

$$Kx = y \tag{1.1}$$

by the method of regularization, we introduce a nonzero parameter α, a closed densely defined linear operator L from X into Y, and the functional

$$G_\alpha(x) = \| Kx - y \|^2 + \alpha^2 \| Lx \|^2$$

defined on the domain of L. We then seek an element x_α solving the minimization problem

$$G_\alpha(x_\alpha) = \inf_{x \in \mathscr{D}(L)} G_\alpha(x) \tag{1.2}$$

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and show that the x_α converge to a least squares solution x_0 of equation (1.1) as $\alpha \to 0$. The operator L is called a *regularizing operator* or a *regularizer* associated with equation (1.1).

In many applications (1.1) is either ill-posed or numerically unstable, particularly in the presence of noise in data measurements. Historically, regularization was introduced [11–13] as a means to overcome the instability of numerical computation of least squares solutions to such equations, with K arising from a first kind integral equation and L being a simple linear differential operator. Many other useful choices of K and L are now available allowing one to put regularization in statistical and other settings. Readers interested in the numerous applications of regularization are referred to the texts and bibliographies of [10, 14, 1, 2, and 4].

While many theoretical and practical advances have been made in the method of regularization since its emergence as a viable numerical method in the solution of ill-posed problems, a proper mathematical setting has been lacking when the regularizing operator L is a linear differential operator defined in $L^2[a, b]$. The principal difficulty has centered upon the role of boundary values in the regularization process. In practice, boundary values are often artificially imposed leading to numerical solutions which decay severely near the end points of $[a, b]$ while being good approximates elsewhere in the interval. One goal of this paper is to put regularization with a differential operator L in a Hilbert space setting which will dispense with the artificial imposition of boundary constraints, but will also admit (see [7]) efficient numerical computation of regularized solutions by weak least squares finite element schemes together with optimal L^2 and L^∞ error estimates. A second goal is to extend extant existence, uniqueness, and convergence theories to general operators K and L.

We now outline the paper, which is divided into seven sections: mathematical preliminaries (Section 2), existence and uniqueness analyses (Section 3), convergence analysis (Section 4), regularization as a least squares process (Section 3), applications to first kind integral equations (Section 6), and representation theory (Section 7). Sections 2 through 5 apply to general linear operators K and L satisfying the conditions: (I) $\mathcal{N}(K) \cap \mathcal{N}(L) = \{0\}$, (II) the range $\mathcal{R}(L)$ is closed, and (III) K is invertible on the null space $\mathcal{N}(L)$. Section 5 follows Nashed [8] and introduces the linear operator $T_\alpha x := (Kx, \alpha Lx)$ from the domain $\mathcal{D}(L)$ into the produce space $Y \oplus Y$, and in this setting regularization becomes a least squares process. In Sections 6 and 7 we specialize to the case when K is a first kind integral operator and L is an nth order differential operator in $L^2[a, b]$. Section 7 contains a generalized Green's function representation for the generalized inverse of T_α; this representation will play a central role in [7] for obtaining spline approximates to the solution x_α of the minimization problem (1.2).

2. Mathematical Preliminaries

In this section we introduce the various spaces and operators used throughout the paper and summarize some basic properties of least squares solutions of equation (1.1). Let X and Y be Hilbert spaces with the inner product $(\, , \,)$ and norm $\| \; \|$ denoted by the same symbols. Let K be a bounded everywhere defined linear operator from X into Y, so the adjoint operator K^* is a bounded everywhere defined linear operator from Y into X satisfying

$$(Kx, y) = (x, K^*y) \qquad \text{for all } x \in X, \quad y \in Y.$$

Let P and $I - P$ be the orthogonal projections from Y onto $\overline{\mathscr{R}(K)}$ and $\overline{\mathscr{R}(K)}^{\perp} = \mathscr{R}(K)^{\perp} = \mathscr{N}(K^*)$, respectively, let $\tilde{K} = K \mid \mathscr{N}(K)^{\perp}$, and let $K^{\dagger}$ be the generalized inverse of K:

$$\mathscr{D}(K^{\dagger}) = \mathscr{R}(K) + \mathscr{R}(K)^{\perp}, \qquad K^{\dagger}y = \tilde{K}^{-1}Py.$$

The set $K^{\dagger}y + \mathscr{N}(K)$ is the set of all least squares solutions of equation (1.1), and $K^{\dagger}y$ is the least squares solution having minimal norm.

The following theorem is well known.

THEOREM 2.1. *If $y \in Y$ with $y = \bar{y} + z$ where $\bar{y} \in \overline{\mathscr{R}(K)}$ and $z \in \mathscr{R}(K)^{\perp}$, then the following statements are equivalent:*

(i) $y \in \mathscr{D}(K^{\dagger}) = \mathscr{R}(K) + \mathscr{R}(K)^{\perp}$.

(ii) $\bar{y} \in \mathscr{R}(K)$.

(iii) *There exists $x \in X$ that is a least squares solution of equation* (1.1):

$$\| Kx - y \| = \inf_{u \in X} \| Ku - y \| . \tag{2.1}$$

(iv) *There exists $x \in X$ satisfying the normal equation:*

$$K^*Kx = K^*y. \tag{2.2}$$

(v) *There exists $x \in X$ satisfying the projected equation:*

$$Kx = \bar{y}. \tag{2.3}$$

Moreover, if anyone of these statements holds, then any element $x \in X$ which satisfies either (2.1), (2.2), *or* (2.3) *automatically satisfies the other two.*

Next, we introduce the regularizing operator. Let L be a closed densely defined linear operator from X into Y, i.e., L is a linear operator with domain $\mathscr{D}(L)$ and range $\mathscr{R}(L)$ which are subspaces of X and Y, respectively, $\mathscr{D}(L)$ is dense in X, and the graph $\Gamma(L)$ is closed in the product space $X \oplus Y$. We will frequently work with $\mathscr{D}(L)$ under the *graph norm structure:*

$$(x, y)_L = (x, y) + (Lx, Ly), \qquad \|x\|_L = (x, x)_L^{1/2}.$$

Since L is a closed operator, it follows that $\mathscr{D}(L)$ is a Hilbert space under this structure.

The adjoint operator L^* is now a closed densely defined linear operator from Y into X with domain $\mathscr{D}(L^*)$ consisting of all $y \in Y$ such that there exists $y^* \in X$ satisfying $(Lx, y) = (x, y^*)$ for all $x \in \mathscr{D}(L)$ and with $L^*y = y^*$. Clearly

$$(Lx, y) = (x, L^*y) \qquad \text{for all } x \in \mathscr{D}(L), \quad y \in \mathscr{D}(L^*).$$

We also have need of the product operator L^*L from X into X given by

$$\mathscr{D}(L^*L) = \{x \in \mathscr{D}(L) \mid Lx \in \mathscr{D}(L^*)\}, \qquad L^*Lx = L^*(Lx),$$

and it is well known that L^*L is self-adjoint.

Let P_0 and $I - P_0$ be the orthogonal projections from Y onto $\overline{\mathscr{R}(L)}$ and $\overline{\mathscr{R}(L)}^\perp = \mathscr{R}(L)^\perp = \mathscr{N}(L^*)$, respectively, let $\bar{L} = L \mid \mathscr{D}(L) \cap \mathscr{N}(L)^\perp$, and let $L^\dagger$ be the generalized inverse of L:

$$\mathscr{D}(L^\dagger) = \mathscr{R}(L) + \mathscr{R}(L)^\perp, \qquad L^\dagger y = \bar{L}^{-1}P_0 y.$$

We observe that if $\mathscr{R}(L)$ is closed, then $\mathscr{D}(L^\dagger) = Y$ and $L^\dagger$ is continuous from Y under its given topology into $\mathscr{D}(L)$ under the graph norm topology. Also, in the same manner we can introduce the generalized inverses of L^* and L^*L, and

$$(L^*)^\dagger = (L^\dagger)^*, \qquad (L^*L)^\dagger = L^\dagger(L^*)^\dagger.$$

For a given element $y \in Y$ and for each real number $\alpha \neq 0$, let G_α be the functional defined on $\mathscr{D}(L)$ by

$$G_\alpha(x) = \|Kx - y\|^2 + \alpha^2 \|Lx\|^2, \qquad x \in \mathscr{D}(L).$$

In the next section we will introduce conditions on the linear operators K and L which guarantee that the minimization problem (1.2) has a unique solution $x_\alpha \in \mathscr{D}(L)$, and in addition, we show that if $y \in \mathscr{R}(K \mid \mathscr{D}(L)) + \mathscr{N}(K^*)$, then the x_α converge in the graph norm as $\alpha \to 0$ to an element $x_0 \in \mathscr{D}(L)$ which is a least squares solution of equation (1.1). In general, we do not have $x_0 = K^\dagger y$, although this is true in the special case $X = Y$, $L = I$. A precise characterization of the limit element x_0 will be given in Section 4.

3. Existence and Uniqueness Analyses

We begin our analysis with a theorem which characterizes an element x_α minimizing G_α as a solution of the so-called *regularized normal equation* (see (3.1) below). Although the essential features of this theorem are well known, it should be emphasized that the original minimization problem (1.2) is set in the domain of L, while (3.1) is set in the domain of L^*L. Consequently, when L is an nth order differential operator in $L^2[a, b]$, Theorem 3.1 guarantees that the x_α belongs to the Sobolev space $H^{2n}[a, b]$ instead of just $H^n[a, b]$. This super-regularity of the x_α will be very important in our subsequent numerical work [7].

THEOREM 3.1. *The functional G_α has a minimum at a point $x \in \mathscr{D}(L)$ iff $x \in \mathscr{D}(L^*L)$ and*

$$K^*Kx + \alpha^2 L^*Lx = K^*y. \tag{3.1}$$

Proof. Take elements x and w in $\mathscr{D}(L)$ and form the quadratic polynomial $\phi(\lambda) = G_\alpha(x + \lambda w)$. If G_α achieves a minimum at x, then ϕ has a minimum at $\lambda = 0$, and hence,

$$0 = \phi'(0) = 2(Kx - y, Kw) + 2\alpha^2(Lx, Lw)$$

or

$$(Lw, Lx) = (w, -(1/\alpha^2)\,[K^*Kx - K^*y]).$$

Since this holds for all $w \in \mathscr{D}(L)$, we conclude that $Lx \in \mathscr{D}(L^*)$ or $x \in \mathscr{D}(L^*L)$, and $L^*Lx = -(1/\alpha^2)\,[K^*Kx - K^*y]$, establishing (3.1).

Conversely, suppose $x \in \mathscr{D}(L^*L) \subseteq \mathscr{D}(L)$ satisfies (3.1). Take any $u \in \mathscr{D}(L)$, set $w = u - x$, and form the polynomial $\phi(\lambda)$ as above. Then $\phi'(0) = 0$, so ϕ has a minimum at $\lambda = 0$, and hence, $\phi(0) \leqslant \phi(1)$ or $G_\alpha(x) \leqslant G_\alpha(u)$. Q.E.D.

We observe that if $x \in \mathscr{N}(K^*K + \alpha^2 L^*L)$, then

$$0 = (K^*Kx + \alpha^2 L^*Lx, x) = \|Kx\|^2 + \alpha^2 \|Lx\|^2,$$

or $x \in \mathscr{N}(K) \cap \mathscr{N}(L)$. Thus,

$$\mathscr{N}(K^*K + \alpha^2 L^*L) = \mathscr{N}(K) \cap \mathscr{N}(L),$$

and from Theorem 3.1 we see that the minimization problem (1.2) has uniqueness iff

$$\mathscr{N}(K) \cap \mathscr{N}(L) = \{0\}.$$

Also, by the theorem a sufficient condition to guarantee both existence and uniqueness for the minimization problem (1.2) is that the linear operator $K^*K + \alpha^2 L^*L$ be invertible. We now introduce conditions which guarantee this.

Throughout the sequel we assume that K and L satisfy the following conditions:

(I) $\mathscr{N}(K) \cap \mathscr{N}(L) = \{0\}$.

(II) $\mathscr{R}(L)$ is closed.

(III) There exists $\beta > 0$ such that

$$\| Kx \| \geq \beta \| x \| \qquad \text{for all } x \in \mathscr{N}(L). \tag{*}$$

These conditions are similar to those used by Nashed [8] for bounded regularizers L.

Remark 3.2. If $\mathscr{N}(K) \cap \mathscr{N}(L) = \{0\}$ and $\dim \mathscr{N}(L) < \infty$, then condition (III) holds automatically. This will be the situation in our applications when K is an integral operator and L is a differential operator in $L^2[a, b]$: only condition (I) must be assumed and the other two follow automatically.

Under conditions (I)–(III) we proceed to establish the existence and uniqueness of the element x_α solving the minimization problem (1.2). Toward this end we introduce a second structure for the subspace $\mathscr{D}(L)$:

$$(x, y)_* = (Kx, Ky) + (Lx, Ly), \qquad \| x \|_* = (x, x)_*^{1/2}.$$

We will refer to this new structure as the **-structure* on $\mathscr{D}(L)$. The next lemma gives its basic properties.

LEMMA 3.3.

(i) $(\ ,\ )_*$ *is an inner product on* $\mathscr{D}(L)$.

(ii) $\mathscr{D}(L)$ *is a Hilbert space under the inner product* $(\ ,\ )_*$.

(iii) $\| \ \|_L$ *and* $\| \ \|_*$ *are equivalent norms on* $\mathscr{D}(L)$.

Proof. Part (i) is trivial since $(\ ,\ )_*$ is clearly a symmetric bilinear form on $\mathscr{D}(L)$, and it is positive definite by condition (I). Suppose x_i, $i = 1, 2,...$, is a sequence in $\mathscr{D}(L)$ with

$$\| x_i - x_j \|_*^2 = \| Kx_i - Kx_j \|^2 + \| Lx_i - Lx_j \|^2 \to 0$$

as $i, j \to \infty$. Then the Kx_i and the Lx_i both form Cauchy sequences in Y, and hence, there exist elements ξ and η in Y with

$$Kx_i \to \xi, \qquad Lx_i \to \eta \qquad \text{as} \qquad i \to \infty.$$

Writing $x_i = u_i + v_i$ with $u_i \in \mathscr{N}(L)$ and $v_i \in \mathscr{D}(L) \cap \mathscr{N}(L)^\perp$, we have $Lx_i = Lv_i \to \eta$, and since $\mathscr{R}(L)$ is closed, it follows that the generalized inverse $L^\dagger$ is continuous and $\eta \in \mathscr{R}(L)$, and hence,

$$v_i = L^\dagger L v_i \to L^\dagger \eta \in \mathscr{D}(L) \cap \mathscr{N}(L)^\perp.$$

The continuity of K then yields $Kv_i \to KL^\dagger\eta$, so

$$Ku_i = Kx_i - Kv_i \to \xi - KL^\dagger\eta.$$

On the other hand, by (*) we have

$$\| u_i - u_j \| \leqslant (1/\beta) \| Ku_i - Ku_j \|,$$

which implies that the u_i form a Cauchy sequence in X, and consequently, there exists $u \in \mathcal{N}(L)$ with $u_i \to u$. Finally, if we set $x = u + L^\dagger\eta$, then $x \in \mathscr{D}(L)$ and

$$x_i = u_i + v_i \to u + L^\dagger\eta = x$$

and

$$Kx_i \to Kx = \xi, \qquad Lx_i \to \eta.$$

Thus, because L is closed, it follows that $Lx = \eta$, so

$$\| x_i - x \|_*^2 = \| Kx_i - \xi \|^2 + \| Lx_i - \eta \|^2 \to 0.$$

We conclude that $\mathscr{D}(L)$ is complete under the inner product $(\ , \ )_*$, establishing (ii).

For part (iii) we have

$$\begin{aligned} \| x \|_*^2 &= \| Kx \|^2 + \| Lx \|^2 \\ &\leqslant \| K \|^2 \| x \|^2 + \| Lx \|^2 \\ &\leqslant \max\{\| K \|^2, 1\} \| x \|_L^2 \end{aligned}$$

for all $x \in \mathscr{D}(L)$. This completes the proof. Q.E.D.

Our second lemma yields the invertibility of $K^*K + \alpha^2 L^*L$.

LEMMA 3.4. *For each $\alpha \neq 0$ the linear operator $K^*K + \alpha^2 L^*L$ is self-adjoint and invertible.*

Proof. We know that L^*L and $\alpha^2 L^*L$ are self-adjoint, and K^*K is bounded and self-adjoint with $\mathscr{D}(K^*K) = X$ containing $\mathscr{D}(L^*L)$. Hence, by a theorem of Kato [3, p. 287] it follows that $K^*K + \alpha^2 L^*L$ is self-adjoint.

Using Lemma 3.3, choose $m > 0$ such that

$$\| x \|_* \geqslant m \| x \|_L \qquad \text{for all } x \in \mathscr{D}(L). \tag{3.2}$$

By the Schwarz inequality we have

$$\begin{aligned} \| K^*Kx + \alpha^2 L^*Lx \| \| x \| &\geqslant (K^*Kx + \alpha^2 L^*Lx, x) \\ &= \| Kx \|^2 + \alpha^2 \| Lx \|^2 \\ &\geqslant m^2 \min\{1, \alpha^2\} \| x \|_L^2 \\ &\geqslant m^2 \min\{1, \alpha^2\} \| x \|^2 \end{aligned}$$

or

$$\| K^*Kx + \alpha^2L^*Lx \| \geqslant m^2 \min\{1, \alpha^2\} \| x \| \qquad \text{for all } x \in \mathscr{D}(L^*L). \tag{3.3}$$

From (3.3) we see that $K^*K + \alpha^2L^*L$ is $1 - 1$ and has closed range with

$$\mathscr{R}(K^*K + \alpha^2L^*L) = \mathscr{N}(K^*K + \alpha^2L^*L)^{\perp} = \{0\}^{\perp} = X.$$

Also, $(K^*K + \alpha^2L^*L)^{-1}$ is bounded by (3.3). Q.E.D.

We next state our principal existence and uniqueness theorem, which is an immediate consequence of Theorem 3.1 and Lemma 3.4.

THEOREM 3.5. *If the linear operators K and L satisfy conditions* (I)–(III), *then for each $y \in Y$ and for each $\alpha \neq 0$ there exists a unique element $x_\alpha \in \mathscr{D}(L)$ satisfying*

$$\| Kx_\alpha - y \|^2 + \alpha^2 \| Lx_\alpha \|^2 = \inf_{x \in \mathscr{D}(L)} [\| Kx - y \|^2 + \alpha^2 \| Lx \|^2].$$

Moreover, the element x_α is also characterized as the unique element in $\mathscr{D}(L^*L)$ satisfying

$$K^*Kx_\alpha + \alpha^2L^*Lx_\alpha = K^*y.$$

4. CONVERGENCE ANALYSIS

In this section we prove the convergence of the x_α to a least squares solution x_0 of equation (1.1) provided the y is reasonably situated. It should be emphasized that the convergence is a very strong type of convergence, taking place in the $*$-norm $\| \ \|_*$ or the equivalent graph norm $\| \ \|_L$. Also, we give several characterizations of the limit element x_0, develop error estimates for $x_\alpha - x_0$, and present several examples which illustrate the dependence of the rate of convergence on the parameter α. In most of this material we work with $\mathscr{D}(L)$ under its $*$-structure.

Let

$$A = K \mid \mathscr{D}(L),$$

the restriction of K to $\mathscr{D}(L)$. Clearly A and L are both bounded everywhere defined linear operators from $\mathscr{D}(L)$ under the $*$-structure into Y under its given structure, and hence, there exist adjoint operators $A^\#$ and $L^\#$ which are bounded everywhere defined linear operators from Y under its given structure into $\mathscr{D}(L)$ under the $*$-structure. These adjoints are determined by the conditions

$$(Ax, y) = (x, A^\# y)_*, \qquad (Lx, y) = (x, L^\# y)_*,$$

for all $x \in \mathscr{D}(L)$, $y \in Y$.

In the next lemma we give explicit representations for $A^\#$ on Y and for $L^\#$

on $\mathscr{D}(L^*)$ and prove that $A^\# A + L^\# L = I$ on $\mathscr{D}(L)$. We obtain the latter identity even though we do not have a closed expression for $L^\#$ on Y.

LEMMA 4.1.

(i) $A^\# y = (K^*K + L^*L)^{-1} K^*y$ *for all* $y \in Y$.

(ii) $L^\# y = (K^*K + L^*L)^{-1} L^*y$ *for all* $y \in \mathscr{D}(L^*)$.

(iii) $A^\# Ay + L^\# Ly = Iy$ *for all* $y \in \mathscr{D}(L)$.

Proof. By Lemma 3.4 the linear operator $K^*K + L^*L$ is invertible. Take $x \in \mathscr{D}(L)$ and $y \in Y$, and set $u = (K^*K + L^*L)^{-1} K^*y$. Clearly $u \in \mathscr{D}(L^*L)$, $K^*Ku + L^*Lu = K^*y$, and

$$\begin{aligned}(x, A^\# y)_* &= (Ax, y)\\ &= (x, K^*y)\\ &= (x, K^*Ku + L^*Lu)\\ &= (x, u)_* .\end{aligned}$$

Since this holds for every $x \in \mathscr{D}(L)$, we conclude that $A^\# y = u$. This establishes (i), and the same argument yields (ii).

To obtain (iii) we begin with an element $y \in \mathscr{D}(L^*L)$. Clearly $Ly \in \mathscr{D}(L^*)$, so using (i) and (ii) we have

$$A^\# Ay + L^\# Ly = (K^*K + L^*L)^{-1} K^*Ky + (K^*K + L^*L)^{-1} L^*Ly = Iy$$

or

$$A^\# Ay + L^\# Ly = Iy \qquad \text{for all } y \in \mathscr{D}(L^*L). \tag{4.1}$$

We can extend (4.1) to the larger subspace $\mathscr{D}(L)$ by using a limit argument. Indeed, take any $y \in \mathscr{D}(L)$. It is well known that $\mathscr{D}(L^*L)$ is a *core* of L (see [3, p. 166 and p. 275]), and hence, there exists a sequence $y_i \in \mathscr{D}(L^*L)$, $i = 1, 2,\ldots$, such that $y_i \to y$ and $Ly_i \to Ly$. This implies that $Ay_i \to Ay$, while from (4.1) we have

$$A^\# Ay_i + L^\# Ly_i = Iy_i .$$

The proof is completed by letting $i \to \infty$ in the last equation and by invoking the continuity of $A^\#$ and $L^\#$. Q.E.D.

To establish the convergence of the x_α, let

$$M := \{x \in \mathscr{D}(L) : (x, u)_* = 0 \text{ for all } u \in \mathscr{N}(A)\},$$

let $\bar{A} = A \mid M$, and let Q and $I - Q$ be the orthogonal projections from Y onto $\overline{\mathscr{R}(A)}$ and $\overline{\mathscr{R}(A)}^\perp = \mathscr{R}(A)^\perp = \mathscr{N}(A^\#)$, respectively. Clearly

$$\mathscr{N}(A) = \mathscr{N}(K) \cap \mathscr{D}(L),$$

and M is the orthogonal complement of $\mathcal{N}(A)$ in $\mathscr{D}(L)$ under the inner product $(\ ,\ )_*$ with

$$M = \{x \in \mathscr{D}(L) \mid (Lx, Lu) = 0 \text{ for all } u \in \mathcal{N}(K) \cap \mathscr{D}(L)\}.$$

DEFINITION 4.2. For $\mathscr{D}(L)$ under the inner product $(\ ,\ )_*$ and Y under its given inner product $(\ ,\ )$, the generalized inverse of A relative to these inner products is denoted by $A^\dagger$:

$$\mathscr{D}(A^\dagger) = \mathscr{R}(A) + \mathscr{R}(A)^\perp, \qquad A^\dagger y = A^{-1}Qy.$$

In terms of this generalized inverse we next state our main convergence result.

THEOREM 4.3. *Under the hypothesis of Theorem* 3.5, *the* x_α *converge to some element* $x_0 \in \mathscr{D}(L)$ *as* $\alpha \to 0$ *in the norm* $\|\ \|_*$ *or the equivalent norm* $\|\ \|_L$ *iff*

$$y \in \mathscr{D}(A^\dagger) = \mathscr{R}(A) + \mathscr{R}(A)^\perp = \mathscr{R}(A) + \mathcal{N}(A^*),$$

in which case x_0 *is the least squares solution of* (1.1) *given by*

$$x_0 = \lim_{\alpha \to 0} x_\alpha = A^\dagger y. \tag{4.2}$$

Proof. First, assume $\|x_\alpha - x_0\|_* \to 0$ as $\alpha \to 0$ for some $x_0 \in \mathscr{D}(L)$. By Lemma 3.3 we have $x_\alpha \to x_0$, $Ax_\alpha \to Ax_0$, and $Lx_\alpha \to Lx_0$, and hence, by the continuity of $A^\#$ and $L^\#$ we must have $A^\# A x_\alpha \to A^\# A x_0$ and $L^\# L x_\alpha \to L^\# L x_0$. Now

$$K^*Kx_\alpha + \alpha^2 L^* L x_\alpha = K^* y$$

with $x_\alpha \in \mathscr{D}(L^*L)$ and $Lx_\alpha \in \mathscr{D}(L^*)$, so applying $(K^*K + L^*L)^{-1}$ we obtain

$$A^\# A x_\alpha + \alpha^2 L^\# L x_\alpha = A^\# y. \tag{4.3}$$

If we let $\alpha \to 0$ in (4.3), then we get $A^\# A x_0 = A^\# y$ or $K^*Kx_0 = K^*y$, and it follows by Theorem 2.1 that $y \in \mathscr{R}(A) + \mathscr{R}(A)^\perp$ and x_0 is a least squares solution of (1.1).

Second, assume $y \in \mathscr{R}(A) + \mathscr{R}(A)^\perp$. By substituting Lemma 4.1(iii) into equation (4.3), it can be rewritten as

$$A^\# A x_\alpha + \alpha^2 (Ix_\alpha - A^\# A x_\alpha) = A^\# y$$

or

$$A^\# A x_\alpha + \alpha^2 (1 - \alpha^2)^{-1} I x_\alpha = (1 - \alpha^2)^{-1} A^\# y. \tag{4.4}$$

Therefore,

$$x_\alpha = (1 - \alpha^2)^{-1} [A^\# A + \alpha^2 (1 - \alpha^2)^{-1} I]^{-1} A^\# y \tag{4.5}$$

for $0 < |\alpha| < 1$. But it is well known that

$$\lim_{\beta \to 0^+} (A^{\#}A + \beta I)^{-1} A^{\#} y = A^{\dagger} y \tag{4.6}$$

provided $y \in \mathscr{D}(A^{\dagger}) = \mathscr{R}(A) + \mathscr{R}(A)^{\perp}$ (see [9, p. 386]), and hence, from (4.5) we conclude that $x_\alpha \to A^{\dagger} y$ as $\alpha \to 0$ with the convergence in $\mathscr{D}(L)$ under the $*$-structure. Q.E.D.

Remark 4.4. The convergence in Theorem 4.3 is equivalent to

$$x_\alpha \to x_0, \qquad Kx_\alpha \to Kx_0, \qquad Lx_\alpha \to Lx_0$$

as $\alpha \to 0$. In case L is an nth order differential operator in $L^2[a, b]$, this convergence corresponds to convergence in the Sobolev space $H^n[a, b]$, i.e., uniform convergence on $[a, b]$ of the derivatives of order $0, 1, \ldots, n-1$ and L^2 convergence of the nth derivatives.

Remark 4.5. The condition $y \in \mathscr{D}(A^{\dagger})$ can be rewritten as

$$y \in \mathscr{R}(K \mid \mathscr{D}(L)) + \mathscr{N}(K^*).$$

Remark 4.6. We can give a second characterization of the limit element $x_0 = A^{\dagger} y$ which corresponds to the one given by Nashed [8, pp. 309–311] for bounded regularizers. Let

$$S_y = \{x \in \mathscr{D}(L) \mid K^*Kx = K^*y\},$$

the set of least squares solutions of (1.1) belonging to $\mathscr{D}(L)$. Now in $\mathscr{D}(L)$ the equation $K^*Kx = K^*y$ is equivalent to the equation $A^{\#}Ax = A^{\#}y$, and hence, by Theorem 2.1 $S_y \neq \varnothing$ iff $y \in \mathscr{D}(A^{\dagger})$, in which case

$$S_y = \mathscr{N}(A) + x_0 = \mathscr{N}(K) \cap \mathscr{D}(L) + x_0 .$$

Suppose $y \in \mathscr{D}(A^{\dagger})$, and take any $x \in S_y$, say $x = u + x_0$ where $u \in \mathscr{N}(K) \cap \mathscr{D}(L)$. Then $x_0 = A^{\dagger} y$, $Kx_0 = Kx$, and from the definition of $A^{\dagger}$ we have

$$\|x_0\|_*^2 = \|Kx_0\|^2 + \|Lx_0\|^2 \leqslant \|x\|_*^2 = \|Kx_0\|^2 + \|Lx\|^2$$

or

$$\|Lx_0\| \leqslant \|Lx\| \qquad \text{for all } x \in S_y . \tag{4.7}$$

Thus, x_0 is the element in S_y which minimizes the quantity $\|Lx\|$ over S_y.

Our convergence analysis has been based on the well-known result (4.6). However, it is possible to establish the convergence directly by deriving a suitable error estimate for the quantity $x_\alpha - x_0$. This has been done by Ivanov [2] in the special case $L = I$, and we now proceed to generalize his results to the case of arbitrary L.

Assuming $y \in \mathscr{D}(A^{\#})$, we know that

$$K^*Kx_\alpha + \alpha^2 L^*Lx_\alpha = K^*y \qquad \text{and} \qquad K^*Kx_0 = K^*y$$

with $x_0 \in M \subseteq \mathscr{D}(L)$. Therefore,

$$\begin{aligned}\|Kx_\alpha - Kx_0\|^2 &= (K^*Kx_\alpha - K^*Kx_0, x_\alpha - x_0)\\ &= -\alpha^2(L^*Lx_\alpha, x_\alpha - x_0)\\ &= -\alpha^2(Lx_0, Lx_\alpha - Lx_0) - \alpha^2\|Lx_\alpha - Lx_0\|^2\end{aligned}$$

or

$$\begin{aligned}\|Kx_\alpha - Kx_0\|^2 = -\alpha^2(x_0, x_\alpha - x_0)_* &- \alpha^2\|x_\alpha - x_0\|_*^2\\ &+ \alpha^2(Kx_0, Kx_\alpha - Kx_0) + \alpha^2\|Kx_\alpha - Kx_0\|^2.\end{aligned} \tag{4.8}$$

For each $R > 0$ set

$$\delta_R = \inf\{\|x_0 - A^{\#}w\|_* : w \in Y, \|w\| \leqslant R\}. \tag{4.9}$$

Since $x_0 \in M = \overline{\mathscr{R}(A^{\#})}$ (closure in $\mathscr{D}(L)$ under the $*$-structure), it follows that $\delta_R \to 0$ as $R \to \infty$. Take any $R > 0$ and take any $w \in Y$ with $\|w\| \leqslant R$. Then from (4.8) we have

$$\begin{aligned}&\|Kx_\alpha - Kx_0\|^2\\ &\quad= -\alpha^2(x_0 - A^{\#}w, x_\alpha - x_0)_* - \alpha^2(w, Kx_\alpha - Kx_0) - \alpha^2\|x_\alpha - x_0\|_*^2\\ &\qquad + \alpha^2(Kx_0, Kx_\alpha - Kx_0) + \alpha^2\|Kx_\alpha - Kx_0\|^2\\ &\quad\leqslant \alpha^2\|x_0 - A^{\#}w\|_*\|x_\alpha - x_0\|_* + \alpha^2R\|Kx_\alpha - Kx_0\| - \alpha^2\|x_\alpha - x_0\|_*^2\\ &\qquad + \alpha^2\|Kx_0\|\|Kx_\alpha - Kx_0\| + \alpha^2\|Kx_\alpha - Kx_0\|^2,\end{aligned}$$

and hence, taking the infinum over all such w gives

$$\begin{aligned}\|Kx_\alpha - Kx_0\|^2 \leqslant \alpha^2\delta_R\|x_\alpha - x_0\|_* &+ \alpha^2R\|Kx_\alpha - Kx_0\| - \alpha^2\|x_\alpha - x_0\|_*^2\\ &+ \alpha^2\|Kx_0\|\|Kx_\alpha - Kx_0\| + \alpha^2\|Kx_\alpha - Kx_0\|^2.\end{aligned}$$

Completing the square on the last inequality, we get

$$\begin{aligned}(1 - \alpha^2)\,[\|Kx_\alpha - Kx_0\| &- (\alpha^2/2)\,(1 - \alpha^2)^{-1}\,(R + \|Kx_0\|)]^2\\ &+ \alpha^2[\|x_\alpha - x_0\|_* - (\delta_R/2)]^2\\ &\leqslant (\alpha^4/4)\,(1 - \alpha^2)^{-1}\,(R + \|Kx_0\|)^2 + \alpha^2\delta_R^2/4.\end{aligned} \tag{4.10}$$

Finally, from (4.10) we obtain our main error estimate, which we state as

THEOREM 4.7. *Under the hypothesis of Theorem* 3.5, *let* $y \in \mathscr{D}(A^\dagger)$, *let* $x_0 = A^\dagger y$, *and for each* $R > 0$ *let* δ_R *be defined by* (4.9). *Then*

$$\| x_\alpha - x_0 \|_* \leqslant \delta_R/2 + \tfrac{1}{2}[\alpha^2(1 - \alpha^2)^{-1}(R + \| Kx_0 \|)^2 + \delta_R{}^2]^{1/2}. \qquad (4.11)$$

Using (4.11) a simple limit argument shows that $\| x_\alpha - x_0 \|_* \to 0$ as $\alpha \to 0$. In general, the rate of convergence $r(\alpha) = \| x_\alpha - x_0 \|_*$ can vary with the problem at hand. We illustrate this with several examples.

EXAMPLE 4.8. Let $x_0 \in \mathscr{R}(A^\#)$, so $x_0 = A^\# w$ for some $w \in Y$. Choosing $R = \| w \|$, we have $\delta_R = 0$, and (4.11) simplifies to

$$\| x_\alpha - x_0 \|_* \leqslant (\alpha/2)(1 - \alpha^2)^{-1/2}(\| w \| + \| Kx_0 \|) = O(\alpha). \qquad (4.12)$$

It can be shown that $x_0 \in \mathscr{R}(A^\#)$ iff $y \in \mathscr{R}(AA^\#) + \mathscr{N}(A^\#)$ iff

$$y \in \mathscr{R}(K(K^*K + L^*L)^{-1}K^*) + \mathscr{N}(K^*),$$

and in case $X = Y$ and $L = I$, this last condition reduces to $y \in \mathscr{R}(KK^*) + \mathscr{N}(K^*)$.

EXAMPLE 4.9. Consider the special case when $X = Y$ and $L = I$ with $\mathscr{R}(K)$ closed. In this situation we have

$$\mathscr{R}(K^*K) = \mathscr{R}(K^*) \qquad \text{and} \qquad \mathscr{N}(K^*K) = \mathscr{N}(K),$$

and the generalized inverse

$$T = (K^*K)^\dagger = K^\dagger(K^*)^\dagger$$

is a bounded everywhere defined linear operator from X into X. Let P_1 and P_2 be the orthogonal projections from X onto $\mathscr{N}(K)$ and $\mathscr{N}(K^*)$, respectively. Now we know that

$$K^*Kx_\alpha + \alpha^2 x_\alpha = K^*y,$$

so applying P_1 gives $\alpha^2 P_1 x_\alpha = 0$, and hence, $P_1 x_\alpha = 0$ or $x_\alpha \in \mathscr{N}(K^*K)^\perp$ for all $\alpha \neq 0$. Applying T gives

$$x_\alpha + \alpha^2 T x_\alpha = TK^*y$$

where

$$TK^*y = K^\dagger(K^*)^\dagger K^*y = K^\dagger(I - P_2)\, y = K^\dagger y,$$

and hence,

$$x_\alpha + \alpha^2 T x_\alpha = K^\dagger y. \tag{4.13}$$

From this last equation we have

$$\|x_\alpha\| \leqslant \|K^\dagger y\| + \alpha^2 \|T\| \|x_\alpha\| \leqslant \|K^\dagger y\| + \tfrac{1}{2}\|x_\alpha\|$$

for all α such that $0 < |\alpha| < (2\|T\|)^{-1/2}$, so the x_α are bounded, and we conclude that

$$\|x_\alpha - K^\dagger y\| = \alpha^2 \|Tx_\alpha\| \leqslant M \|T\| \alpha^2 = 0(\alpha^2). \tag{4.14}$$

This rate of convergence is optimal in this special case, which can be seen by taking $K = L = I$.

EXAMPLE 4.10. Suppose T is an invertible nth order differential operator in $L^2[a, b]$, so T^* is also an invertible nth order differential operator in $L^2[a, b]$. Let $K = T^{-1}$, so $K^* = (T^*)^{-1}$.

Clearly K and K^* are integral operators in $L^2[a, b]$ whose kernels are just the appropriate Green's functions, and K and K^* are both $1-1$ with $\mathscr{R}(K) = \mathscr{D}(T)$ and $\mathscr{R}(K^*) = \mathscr{D}(T^*)$.

Suppose we regularize with $L = I$. We note that $(K^*K)^\dagger = TT^*$, and assuming $y \in \mathscr{R}(K) + \mathscr{R}(K)^\perp = \mathscr{D}(T)$, the equation

$$K^*Kx_\alpha + \alpha^2 x_\alpha = K^*y$$

transforms to

$$x_\alpha + \alpha^2 TT^* x_\alpha = TT^*K^*y = Ty$$

or

$$TT^*x_\alpha + (1/\alpha^2)\, x_\alpha = (1/\alpha^2)\, Ty. \tag{4.15}$$

If we set $x_0 = A^{-1}y = Ty$, then we know that $x_\alpha \to x_0$ as $\alpha \to 0$, and we can examine the rate of convergence.

As a model consider the differential operator T in $L^2[0, 1]$ given by

$$\mathscr{D}(T) = \{x \in H^1[0, 1] \mid x(0) = 0\}, \qquad Tx = x',$$

so T^* and TT^* are the differential operators given by

$$\mathscr{D}(T^*) = \{x \in H^1[0, 1] \mid x(1) = 0\}, \qquad T^*x = -x',$$

and

$$\mathscr{D}(TT^*) = \{x \in H^2[0, 1] \mid x(1) = x'(0) = 0\}, \qquad TT^*x = -x''.$$

For $Ty(t) = x_0(t) = 1$ the solution of (4.15) is easily seen to be

$$x_\alpha(t) = 1 - \cosh^{-1}(1/\alpha) \cosh(t/\alpha).$$

Thus,

$$\| x_\alpha - x_0 \|^2 = \cosh^{-2}(1/\alpha) \int_0^1 \cosh^2(t/\alpha)\, dt = | \alpha/2 |\, | \tanh(1/\alpha)| + \tfrac{1}{2} \cosh^{-2}(1/\alpha)$$

or

$$\| x_\alpha - x_0 \| = O(| \alpha |^{1/2}).$$

5. Regularization as a Least Squares Process

The method of regularization developed in Sections 3 and 4 can also be viewed as the method of least squares by placing it in the right setting. This approach has been outlined by Nashed in [8] for the case of a bounded regularizing operator L. When K is an integral operator and L is an nth order differential operator in $L^2[a, b]$, then the least squares approach to regularization enables one to work in the Sobolev space $H^n[a, b]$ rather than the space $H^{2n}[a, b]$ required by the regularized normal equation (3.1), and this leads to the practical numerical solution of equation (3.1) in terms of weak least squares approximates to x_α from the space $H^n[a, b]$. Such approximates will be developed in complete detail in [7] using the results of this paper.

For the product space $Y \oplus Y$ we introduce the standard inner product $\langle , \rangle$ and norm $| \ |$:

$$\langle (x, y), (u, v) \rangle = (x, u) + (y, v), \qquad |(x, y)| = [\| x \|^2 + \| y \|^2]^{1/2}.$$

For each real number $\alpha \neq 0$ let T_α be the linear operator from X into $Y \oplus Y$ defined by

$$\mathscr{D}(T_\alpha) = \mathscr{D}(L), \qquad T_\alpha x = (Kx, \alpha Lx).$$

Clearly T_α is closed and densely defined with

$$\mathscr{N}(T_\alpha) = \mathscr{N}(K) \cap \mathscr{N}(L),$$

and for $x \in \mathscr{D}(T_\alpha)$ and $z = (\xi, \eta) \in Y \oplus Y$ we have

$$| T_\alpha x - z |^2 = \| Kx - \xi \|^2 + \| \alpha Lx - \eta \|^2.$$

In particular, if $y \in Y$ and if we set $\hat{y} = (y, 0) \in Y \oplus Y$, then

$$| T_\alpha x - \hat{y} |^2 = \| Kx - y \|^2 + \alpha^2 \| Lx \|^2 = G_\alpha(x) \tag{5.1}$$

for all $x \in \mathscr{D}(T_\alpha) = \mathscr{D}(L)$. Also, a straight forward argument shows that the adjoint operator T_α^* is given by

$$\mathscr{D}(T_\alpha^*) = \{(\xi, \eta) \mid \xi \in Y, \eta \in \mathscr{D}(L^*)\}, \qquad T_\alpha^*(\xi, \eta) = K^*\xi + \alpha L^*\eta,$$

while the product operator $T_\alpha^* T_\alpha$ is given by

$$\mathscr{D}(T_\alpha^* T_\alpha) = \mathscr{D}(L^*L), \qquad T_\alpha^* T_\alpha x = K^*Kx + \alpha^2 L^*Lx.$$

LEMMA 5.1. *If the linear operators K and L satisfy conditions* (I)–(III), *then for each $\alpha \neq 0$ the range $\mathscr{R}(T_\alpha)$ is closed in $Y \oplus Y$.*

Proof. Suppose x_i, $i = 1, 2,\ldots$, is a sequence in $\mathscr{D}(T_\alpha)$ with $T_\alpha x_i \to (\xi, \eta)$, so that $Kx_i \to \xi$ and $\alpha L x_i \to \eta$. The proof is easily completed by following the proof of Lemma 3.3 verbatim. Q.E.D.

For each $\alpha \neq 0$ let Q_α be the orthogonal projection from $Y \oplus Y$ onto $\overline{\mathscr{R}(T_\alpha)}$, let $\tilde{T}_\alpha = T_\alpha \mid \mathscr{D}(T_\alpha) \cap \mathscr{N}(T_\alpha)^\perp$, and let $T_\alpha^\dagger$ be the generalized inverse of T_α:

$$\mathscr{D}(T_\alpha^\dagger) = \mathscr{R}(T_\alpha) + \mathscr{R}(T_\alpha)^\perp, \qquad T_\alpha^\dagger z = \tilde{T}_\alpha^{-1} Q_\alpha z.$$

In the next theorem we present the main results for the least squares approach to regularization; the theorem is an immediate consequence of (5.1) and Lemma 5.1.

THEOREM 5.2. *If the linear operators K and L satisfy conditions* (I)–(III), *then the following statements hold:*

(i) *T_α is $1 - 1$, $\mathscr{R}(T_\alpha)$ is closed, and $T_\alpha^\dagger$ is defined on all of $Y \oplus Y$ and is bounded with $T_\alpha^\dagger = \tilde{T}_\alpha^{-1} Q_\alpha$.*

(ii) *For each $y \in Y$ and for each $\alpha \neq 0$ there exists a unique element $x_\alpha \in \mathscr{D}(L)$ satisfying*

$$\| Kx_\alpha - y \|^2 + \alpha^2 \| Lx_\alpha \|^2 = \inf_{x \in \mathscr{D}(L)} [\| Kx - y \|^2 + \alpha^2 \| Lx \|^2].$$

(iii) *The element x_α in part* (ii) *is also characterized as the unique least squares solution of the equation*

$$T_\alpha x = \hat{y} \tag{5.2}$$

where $\hat{y} = (y, 0)$, i.e., $x_\alpha = T_\alpha^\dagger \hat{y}$.

We can relate the ideas of Sections 3 and 4 to those in this section by introducing the normal equation for (5.2):

$$T_\alpha^* T_\alpha x = T_\alpha^* \hat{y}. \tag{5.3}$$

We know that

$$T_\alpha^* T_\alpha x = K^*Kx + \alpha^2 L^*Lx$$

for all $x \in \mathscr{D}(T_\alpha^* T_\alpha) = \mathscr{D}(L^*L)$, and for $\hat{y} = (y, 0) \in \mathscr{D}(T_\alpha^*)$ we have

$$T_\alpha^* \hat{y} = K^*y.$$

Thus, the normal equation (5.3) can be written as

$$K^*Kx + \alpha^2 L^*Lx = K^*y,$$

which is just the regularized normal equation (3.1).

We conclude this section by characterizing the null space of the operator T_α^*. This result will be needed in Section 7 and does not require conditions (I)–(III).

LEMMA 5.3. *An element $(\xi, \eta) \in Y \oplus Y$ belongs to the null space $\mathcal{N}(T_\alpha^*)$ iff $K^*\xi \in \mathcal{R}(L^*)$, $\eta \in \mathcal{D}(L^*)$, and*

$$\eta = -(1/\alpha)(L^*)^\dagger K^*\xi + \eta^* \qquad \textit{for some } \eta^* \in \mathcal{N}(L^*). \tag{5.4}$$

Proof. Let P_3 be the orthogonal projection from Y onto $\mathcal{N}(L^*)$, and let P_4 be the orthogonal projection from X onto $\mathcal{N}(L)$. We know that the generalized inverse $(L^*)^\dagger$ is defined on $\mathcal{R}(L^*) + \mathcal{R}(L^*)^\perp$ with values in $\mathcal{D}(L^*) \cap \mathcal{N}(L^*)^\perp$. Suppose $(\xi, \eta) \in \mathcal{N}(T_\alpha^*)$. Then from our description of T_α^* we have $\eta \in \mathcal{D}(L^*)$ and

$$0 = T_\alpha^*(\xi, \eta) = K^*\xi + \alpha L^*\eta,$$

implying that $K^*\xi \in \mathcal{R}(L^*) \subseteq \mathcal{D}((L^*)^\dagger)$. If we apply $(L^*)^\dagger$ to the last equation, then

$$0 = (L^*)^\dagger K^*\xi + \alpha[\eta - P_3\eta],$$

so η satisfies (5.4) with $\eta^* = P_3\eta \in \mathcal{N}(L^*)$.

On the other hand, suppose $K^*\xi \in \mathcal{R}(L^*)$, $\eta \in \mathcal{D}(L^*)$, and η satisfies (5.4). Since $\mathcal{R}(L^*) \subseteq \mathcal{N}(L)^\perp$, we have $P_4K^*\xi = 0$, while from (5.4) we have

$$\begin{aligned} L^*\eta &= -(1/\alpha) L^*(L^*)^\dagger K^*\xi + 0 \\ &= -(1/\alpha)[K^*\xi - P_4K^*\xi] \\ &= -(1/\alpha) K^*\xi. \end{aligned}$$

Thus,

$$T_\alpha^*(\xi, \eta) = K^*\xi + \alpha L^*\eta = 0.$$

Q.E.D.

6. APPLICATIONS TO FIRST KIND INTEGRAL EQUATIONS

In applying the results of the previous sections to first kind integral equations, we will work in the Hilbert space $L^2[a, b]$ under its standard inner product

$$(x, y) = \int_a^b x(t)\, y(t)\, dt$$

and norm $\|x\| = (x, x)^{1/2}$. Let K be an integral operator of the form

$$Kx(t) = \int_a^b k(t, s)\, x(s)\, ds, \qquad a \leqslant t \leqslant b,$$

mapping $L^2[a, b]$ continuously into $L^2[a, b]$. Given $y \in L^2[a, b]$ we consider the associated equation (1.1):

$$Kx = y,$$

which is an integral equation of the first kind.

To utilize the method of regularization, let L be an nth order differential operator in $L^2[a, b]$ determined by a formal differential operator τ having coefficients in $C^\infty[a, b]$ and by a set of k $(0 \leqslant k \leqslant 2n)$ linearly independent boundary values $B_1, ..., B_k$:

$$\mathscr{D}(L) = \{x \in H^n[a, b] \mid B_i(x) = 0, i = 1,..., k\}, \qquad Lx = \tau x.$$

We know that the adjoint operator L^* is also an nth order differential operator in $L^2[a, b]$, which is determined by the formal adjoint τ^* and by a set of $2n - k$ linearly independent adjoint boundary values $B_1^*,..., B_{2n-k}^*$:

$$\mathscr{D}(L^*) = \{x \in H^n[a, b] \mid B_i^*(x) = 0, i = 1,..., 2n - k\}, \qquad L^*x = \tau^*x.$$

Also, the product operator L^*L is the $2n$th order differential operator in $L^2[a, b]$ given by

$$\begin{aligned}\mathscr{D}(L^*L) = \{x \in H^{2n}[a, b] \mid B_i(x) = 0, i = 1,..., k \text{ and}\\ B_j^*(\tau x) = 0, j = 1,..., 2n - k\},\end{aligned}$$
$$L^*Lx = \tau^*\tau x$$

(see [5]). It is well known that L is a closed densely defined linear operator from $L^2[a, b]$ into $L^2[a, b]$ with $\dim \mathscr{N}(L) \leqslant n < \infty$ and with $\mathscr{R}(L)$ closed in $L^2[a, b]$.

Henceforth, we assume that

$$\mathscr{N}(K) \cap \mathscr{N}(L) = \{0\}.$$

This implies that conditions (I)–(III) are satisfied for the linear operators K and L, and the results of the previous sections can be applied. Let us summarize these results as a theorem (see Theorems 3.5, 4.3, and 5.2).

THEOREM 6.1. *Let K be an integral operator and let L be an* nth *order differential operator in $L^2[a, b]$. If $\mathscr{N}(K) \cap \mathscr{N}(L) = \{0\}$, then the following statements hold:*

(i) *For each* $y \in L^2[a, b]$ *and for each* $\alpha \neq 0$ *there exists a unique element* $x_\alpha \in \mathscr{D}(L)$ *satisfying*

$$\| Kx_\alpha - y \|^2 + \alpha^2 \| Lx_\alpha \|^2 = \inf_{x \in \mathscr{D}(L)} [\| Kx - y \|^2 + \alpha^2 \| Lx \|^2].$$

(ii) *The element* x_α *in part* (i) *is characterized as the unique element* $x_\alpha \in \mathscr{D}(L^*L)$ *satisfying*

$$K^*Kx_\alpha + \alpha^2 L^*Lx_\alpha = K^*y.$$

(iii) *Setting* $\hat{y} = (y, 0) \in L^2[a, b] \oplus L^2[a, b]$, *the element* x_α *in part* (i) *is also characterized as the unique element* $x_\alpha \in \mathscr{D}(T_\alpha) = \mathscr{D}(L)$ *satisfying*

$$\| T_\alpha x_\alpha - \hat{y} \| = \inf_{x \in \mathscr{D}(L)} \| T_\alpha x - \hat{y} \| .$$

(iv) *The* x_α *converge in the* H^n*-Sobolev topology to an element* $x_0 \in \mathscr{D}(L)$ *as* $\alpha \to 0$ *iff*

$$y \in \mathscr{R}(K \mid \mathscr{D}(L)) + \mathscr{N}(K^*),$$

in which case the limit element x_0 *is a least squares solution of the integral equation* $Kx = y$ *and is given precisely by* $x_0 = A^\dagger y$, *where* $A^\dagger$ *is the generalized inverse of* $A = K \mid \mathscr{D}(L)$ *with* $\mathscr{D}(L)$ *under the inner product* $(x, y)_* = (Kx, Ky) + (Lx, Ly)$ *and* $L^2[a, b]$ *under its standard inner product* (x, y).

7. Representation Theory

In this final section we develop a generalized Green's function representation of the generalized inverse $T_\alpha^\dagger$ for fixed $\alpha \neq 0$, where $T_\alpha x = (Kx, \alpha Lx)$ (see Section 5) and where K and L are respectively an integral and an nth order differential operator in $L^2[a, b]$ as in Section 6. This representation will play a key role in the error analysis of [7]. In particular, it will enable us to establish optimal order convergence rates and superconvergence at knots for spline approximates to $x_\alpha = T_\alpha^\dagger \hat{y}$ where $\hat{y} = (y, 0)$.

Throughout this section we fix $\alpha \neq 0$ and assume that

$$\mathscr{N}(K) \cap \mathscr{N}(L) = \{0\},$$

so Theorem 5.2 and Theorem 6.1 are both applicable. Specifically, we know that T_α is $1 - 1$, $\mathscr{R}(T_\alpha)$ is closed in $L^2[a, b] \oplus L^2[a, b]$, and

$$T_\alpha^\dagger = T_\alpha^{-1} Q_\alpha$$

is defined on all of $L^2[a, b] \oplus L^2[a, b]$. Clearly T_α is continuous from $\mathscr{D}(T_\alpha) = \mathscr{D}(L)$ under the H^n-Sobolev topology to $\mathscr{R}(T_\alpha)$ under the L^2-product topology, so $T_\alpha^\dagger$ is continuous in the reverse direction. It should be noted that the H^n-Sobolev topology on $\mathscr{D}(L)$ can be induced by the norm

$$\|x\|_{H^n} = \sum_{i=0}^{n-1} \max_{a\leq t\leq b} |x^{(i)}(t)| + \|x^{(n)}\|$$

or by the equivalent graph norm $\|x\|_L$ (see [5, p. 177]).

For $j = 0, 1,..., n-1$ and for $t \in [a, b]$ we introduce linear functionals $\mu_{j,t}$ and $\lambda_{j,t}$ by

$$\mu_{j,t}(x) = x^{(j)}(t), \qquad x \in \mathscr{D}(T_\alpha) = \mathscr{D}(L),$$

and

$$\lambda_{j,t}(z) = \mu_{j,t}T_\alpha^\dagger(z) = \mu_{j,t}T_\alpha^{-1}Q_\alpha(z), \qquad z \in L^2[a, b] \oplus L^2[a, b].$$

Clearly $\mu_{j,t}$ is continuous on $\mathscr{D}(L)$ under the H^n-topology, and hence, $\lambda_{j,t}$ is continuous on the product space $L^2[a, b] \oplus L^2[a, b]$. Thus, there exists a unique element

$$g_{j,t} = (a_{j,t}, l_{j,t}) \in L^2[a, b] \oplus L^2[a, b] \tag{7.1}$$

such that

$$\lambda_{j,t}(z) = \langle g_{j,t}, z\rangle \qquad \text{for all } z \in L^2[a, b] \oplus L^2[a, b], \tag{7.2}$$

or

$$x^{(j)}(t) = \langle g_{j,t}, z\rangle = (a_{j,t}, \xi) + (l_{j,t}, \eta) \tag{7.3}$$

for all $z = (\xi, \eta) \in L^2[a, b] \oplus L^2[a, b]$ where $x = T_\alpha^\dagger z$.

For $j = 0$ we observe that (7.3) reduces to

$$T_\alpha^\dagger z(t) = \langle g_{0,t}, z\rangle \qquad \text{for all } z \in L^2[a, b] \oplus L^2[a, b].$$

We will refer to equation (7.3) as the *generalized Green's function representation* of $T_\alpha^\dagger$. An immediate consequence of this representation is

LEMMA 7.1. *For $j = 0, 1,..., n-1$ and for $t \in [a, b]$:*

$$x^{(j)}(t) = \langle g_{j,t}, T_\alpha x\rangle = (a_{j,t}, Kx) + \alpha(l_{j,t}, Lx) \tag{7.4}$$

for all $x \in \mathscr{D}(T_\alpha) = \mathscr{D}(L)$.

Note that if $z \in \mathscr{N}(T_\alpha^*)$, then $Q_\alpha z = 0$ and $\lambda_{j,t}(z) = 0$, i.e., $\langle g_{j,t}, z\rangle = 0$ for all $z \in \mathscr{N}(T_\alpha^*)$. This implies that $g_{j,t} \in \mathscr{R}(T_\alpha)$, and hence, there exists a unique element $f_{j,t} \in \mathscr{D}(T_\alpha) = \mathscr{D}(L)$ such that

$$T_\alpha f_{j,t} = g_{j,t}, \tag{7.5}$$

or

$$Kf_{j,t} = a_{j,t}, \qquad \alpha Lf_{j,t} = l_{j,t}, \tag{7.5}$$

for $j = 0, 1,..., n-1$ and for $t \in [a, b]$.

LEMMA 7.2. *$\|g_{j,t}\| \leqslant \|T_\alpha^\dagger\|$ for $j = 0, 1,..., n-1$ and for $t \in [a, b]$, where the norm of $T_\alpha^\dagger$ is taken between the product norm $|\ |$ and the H^n-norm $\|\ \|_{H^n}$.*

Proof. Take $z \in L^2[a, b] \oplus L^2[a, b]$ and set $x = T_\alpha^\dagger z$. Then we have

$$|\lambda_{j,t}(z)| = |x^{(j)}(t)| \leqslant \|x\|_{H^n} = \|T_\alpha^\dagger z\|_{H^n} \leqslant \|T_\alpha^\dagger\| \, |z|,$$

and hence, $\|g_{j,t}\| = \|\lambda_{j,t}\| \leqslant \|T_\alpha^\dagger\|$. Q.E.D.

The basic problem with our generalized Green's function representation (7.3) of $T_\alpha^\dagger$ is that we have very little qualitative information on the representor functions $g_{j,t} = (a_{j,t}, l_{j,t})$ or on the preimage $f_{j,t} = T_\alpha^{-1} g_{j,t}$. We will next establish a second representation for $T_\alpha^\dagger$ on $\mathscr{R}(T_\alpha)$ from which we can obtain definite qualitative information on the representors. This new representation will be based on the generalized Green's function for the differential operator L.

Indeed, the generalized Green's function $G^\dagger(t, s) = G_t^\dagger(s)$ for L is defined as the kernel in the integral representation of the generalized inverse $L^\dagger$:

$$L^\dagger y(t) = \int_a^b G^\dagger(t, s)\, y(s)\, ds, \qquad a \leqslant t \leqslant b, \tag{7.6}$$

for all $y \in L^2[a, b]$. Its basic properties have been developed in [6]. Let us summarize some of the properties that we will need:

(a) If $\phi_1, ..., \phi_{2n}$ and $\psi_1, ..., \psi_{2n}$ are bases for the solution spaces of $\tau\tau^*\phi = 0$ and $\tau^*\tau\psi = 0$, respectively, then there exist unique constants γ_{ij} and γ'_{ij} such that

$$\begin{aligned} G^\dagger(t, s) &= \sum_{i,j=1}^{2n} \gamma_{ij}\psi_i(t)\,\phi_j(s), \qquad a \leqslant s < t \leqslant b, \\ G^\dagger(t, s) &= \sum_{i,j=1}^{2n} \gamma'_{ij}\psi_i(t)\,\phi_j(s), \qquad a \leqslant t < s \leqslant b. \end{aligned} \tag{7.7}$$

(b) $G^\dagger(t, s)$ is infinitely differentiable in both variables for $t \neq s$. In fact,

$$\begin{aligned} \frac{\partial^{p+q} G^\dagger}{\partial t^p\, \partial s^q}(t, s) &= \sum_{i,j=1}^{2n} \gamma_{ij}\psi_i^{(p)}(t)\,\phi_j^{(q)}(s), \qquad a \leqslant s < t \leqslant b, \\ \frac{\partial^{p+q} G^\dagger}{\partial t^p\, \partial s^q}(t, s) &= \sum_{i,j=1}^{2n} \gamma'_{ij}\psi_i^{(p)}(t)\,\phi_j^{(q)}(s), \qquad a \leqslant t < s \leqslant b, \end{aligned} \tag{7.8}$$

for $p, q = 0, 1, 2,...,$ and each of these partial derivatives is bounded independent of t and s.

(c) Fixing $s \in [a, b]$ and considering $G^\dagger(t, s)$ as a function of t, $G^\dagger(t, s)$ is continuous in its derivatives of orders $0, 1,..., n-2$ at the point $t = s$, while its derivatives of orders $n-1, n, \ldots, 2n-1$ have prescribed jump discontinuities at $t = s$.

(d) $G^{\dagger}(t, \cdot) \in \mathscr{N}(L^*)^{\perp} = \mathscr{R}(L)$ for $t \in [a, b]$.

Several of these properties are generalized in the following lemma.

LEMMA 7.3. *If $y \in L^2[a, b]$ and $x = L^{\dagger}y$, then*

$$x^{(j)}(t) = \int_a^b \frac{\partial^j G^{\dagger}}{\partial t^j}(t, s)\, y(s)\, ds \tag{7.9}$$

for $j = 0, 1, \ldots, n-1$ and for $t \in [a, b]$. Also, $(\partial^j G^{\dagger}/\partial t^j)(t, \cdot) \in \mathscr{N}(L^*)^{\perp} = \mathscr{R}(L)$ for $j = 0, 1, \ldots, n-1$ and for $t \in [a, b]$.

Proof. From the regularity of $G^{\dagger}(t, s)$ described above, it follows that we can differentiate under the integral sign in (7.6) to obtain derivatives with respect to t of orders $j = 0, 1, \ldots, n-1$, and this establishes (7.9). For the second part we observe that if $y \in \mathscr{N}(L^*)$, then $L^{\dagger}y = x = 0$, and hence, (7.9) becomes

$$\int_a^b \frac{\partial^j G^{\dagger}}{\partial t^j}(t, s)\, y(s)\, ds = 0 \qquad \text{for all } y \in \mathscr{N}(L^*).$$

Q.E.D.

To establish our new representation, choose a basis $\omega_1, \ldots, \omega_p$ for $\mathscr{N}(L)$ such that $K\omega_1, \ldots, K\omega_p$ is an orthonormal basis for $K[\mathscr{N}(L)]$. Take $z = (\xi, \eta) \in \mathscr{R}(T_\alpha)$ and set $x = T_\alpha^{\dagger}z = T_\alpha^{-1}z$, so $T_\alpha x = z$ or $Kx = \xi$, $\alpha Lx = \eta$. It follows that $\eta \in \mathscr{R}(L)$ and

$$\alpha Lx = \eta = LL^{\dagger}\eta,$$

and hence, $x - (1/\alpha)L^{\dagger}\eta \in \mathscr{N}(L)$ or

$$x = (1/\alpha)L^{\dagger}\eta + \sum_{i=1}^{p} \alpha_i \omega_i .$$

Applying K to this last equation gives

$$Kx = (1/\alpha)\, KL^{\dagger}\eta + \sum_{i=1}^{p} \alpha_i K\omega_i = \xi,$$

and taking the inner product with $K\omega_j$ gives

$$\alpha_j = (\xi, K\omega_j) - (1/\alpha)\,(\eta, (L^*)^{\dagger} K^*K\omega_j).$$

Therefore,

$$x = T_\alpha^{-1}z = (1/\alpha)L^{\dagger}\eta + \sum_{i=1}^{p} (\xi, K\omega_i)\,\omega_i - (1/\alpha)\sum_{i=1}^{p} (\eta, (L^*)^{\dagger} K^*K\omega_i)\,\omega_i \tag{7.10}$$

for all $z = (\xi, \eta) \in \mathscr{R}(T_\alpha)$.

Next, define functions $F(t, s)$ and $G(t, s)$ by

$$F(t, s) = F_t(s) = \sum_{i=1}^{p} \omega_i(t)\, K\omega_i(s), \tag{7.11}$$

$$G(t, s) = G_t(s) = (1/\alpha)\, G^\dagger(t, s) - (1/\alpha) \sum_{i=1}^{p} \omega_i(t)\, (L^*)^\dagger\, K^*K\omega_i(s), \tag{7.12}$$

and for $j = 0, 1,..., n - 1$ set

$$G_j^\dagger(t, s) = G_{j,t}^\dagger(s) = \frac{\partial^j G^\dagger}{\partial t^j}(t, s), \tag{7.13}$$

$$F_j(t, s) = F_{j,t}(s) = \frac{\partial^j F}{\partial t^j}(t, s) = \sum_{i=1}^{p} \omega_i^{(j)}(t)\, K\omega_i(s), \tag{7.14}$$

and

$$\begin{aligned} G_j(t, s) = G_{j,t}(s) &= \frac{\partial^j G}{\partial t^j}(t, s) \\ &= (1/\alpha)\, G_j^\dagger(t, s) - (1/\alpha) \sum_{i=1}^{p} \omega_i^{(j)}(t)\, (L^*)^\dagger\, K^*K\omega_i(s). \end{aligned} \tag{7.15}$$

Upon writing out equation (7.10) it becomes

$$x(t) = T_\alpha^{-1}z(t) = \int_a^b F(t, s)\, \xi(s)\, ds + \int_a^b G(t, s)\, \eta(s)\, ds \tag{7.16}$$

for all $z = (\xi, \eta) \in \mathscr{R}(T_\alpha)$ and for all $t \in [a, b]$. Finally, using Lemma 7.3 we can proceed to differentiate equation (7.16) to obtain

$$x^{(j)}(t) = (F_{j,t}\,, \xi) + (G_{j,t}\,, \eta), \qquad a \leqslant t \leqslant b, \tag{7.17}$$

for all $z = (\xi, \eta) \in \mathscr{R}(T_\alpha)$ where $x = T_\alpha{}^\dagger z = T_\alpha^{-1}z$. This is our second generalized Green's function representation of the linear operator $T_\alpha{}^\dagger$.

It should be emphasized that the first representation (7.3) is valid for all $z \in L^2[a, b] \oplus L^2[a, b]$, while the second representation (7.17) is only valid for $z \in \mathscr{R}(T_\alpha)$. On the other hand in (7.17) we know the representors $F_{j,t}$ and $G_{j,t}$ explicitly by means of the formulas (7.11)–(7.15). Also, an immediate consequence of this new representation is

LEMMA 7.4. *For $j = 0, 1,..., n - 1$ and for $t \in [a, b]$:*

$$x^{(j)}(t) = (F_{j,t}\,, Kx) + \alpha(G_{j,t}\,, Lx) \tag{7.18}$$

for all $x \in \mathscr{D}(T_\alpha) = \mathscr{D}(L)$.

We are now in a position to relate the two generalized Green's function representations of $T_\alpha^\dagger$, and in the process we will obtain useful formulas for the functions $l_{j,t}$ and $f_{j,t}$. We begin with

LEMMA 7.5. $(a_{j,t} - F_{j,t}, l_{j,t} - G_{j,t}) \in \mathscr{N}(T_\alpha^*)$ *for* $j = 0, 1,..., n-1$ *and for* $t \in [a, b]$.

Proof. Fix an integer j with $0 \leqslant j \leqslant n-1$ and fix $t \in [a, b]$. Take any $x \in \mathscr{D}(T_\alpha) = \mathscr{D}(L)$ and set $T_\alpha x = (\xi, \eta)$. From (7.4) and (7.18) we have

$$(a_{j,t}, \xi) + (l_{j,t}, \eta) = x^{(j)}(t) = (F_{j,t}, \xi) + (G_{j,t}, \eta),$$

and hence, $(a_{j,t} - F_{j,t}, \xi) + (l_{j,t} - G_{j,t}, \eta) = 0$ for all $(\xi, \eta) \in \mathscr{R}(T_\alpha)$, i.e., $(a_{j,t} - F_{j,t}, l_{j,t} - G_{j,t}) \in \mathscr{R}(T_\alpha)^\perp = \mathscr{N}(T_\alpha^*)$. Q.E.D.

Combining Lemma 5.3 and Lemma 7.5, we have

$$K^*(a_{j,t} - F_{j,t}) \in \mathscr{R}(L^*), \qquad l_{j,t} - G_{j,t} \in \mathscr{D}(L^*),$$

and

$$l_{j,t} - G_{j,t} = -(1/\alpha)(L^*)^\dagger K^*(a_{j,t} - F_{j,t}) + \eta_{j,t}^* \tag{7.19}$$

for some $\eta_{j,t}^* \in \mathscr{N}(L^*)$. Now observe where the terms in (7.19) lie: $l_{j,t} \in \mathscr{R}(L) = \mathscr{N}(L^*)^\perp$ by (7.5)′, $G_{j,t}^\dagger \in \mathscr{N}(L^*)^\perp$ by Lemma 7.3, $(L^*)^\dagger K^* K\omega_i \in \mathscr{D}(L^*) \cap \mathscr{N}(L^*)^\perp$, $G_{j,t} \in \mathscr{N}(L^*)^\perp$ by (7.15), $(L^*)^\dagger K^*(a_{j,t} - F_{j,t}) \in \mathscr{D}(L^*) \cap \mathscr{N}(L^*)^\perp$, and $\eta_{j,t}^* \in \mathscr{N}(L^*)$. Hence, from (7.19) we conclude that $\eta_{j,t}^* = 0$, i.e.,

$$l_{j,t} = G_{j,t} - (1/\alpha)(L^*)^\dagger K^*(a_{j,t} - F_{j,t}) \tag{7.20}$$

for $j = 0, 1,..., n-1$ and for $t \in [a, b]$.

Recalling that

$$f_{j,t} = T_\alpha^{-1} g_{j,t} = T_\alpha^{-1}(a_{j,t}, l_{j,t}),$$

from equation (7.10) we have

$$f_{j,t} = (1/\alpha) L^\dagger l_{j,t} + \sum_{i=1}^{p} (a_{j,t}, K\omega_i)\, \omega_i - (1/\alpha) \sum_{i=1}^{p} (l_{j,t}, (L^*)^\dagger K^* K\omega_i)\, \omega_i \tag{7.21}$$

for $j = 0, 1,..., n-1$ and for $t \in [a, b]$. If we substitute (7.20) and (7.15) into (7.21), then it can be rewritten as

$$\begin{aligned} f_{j,t} = {} & (1/\alpha^2) L^\dagger G_{j,t}^\dagger - (1/\alpha^2) \sum_{i=1}^{p} \omega_i^{(j)}(t)\,(L^*L)^\dagger K^* K\omega_i \\ & - (1/\alpha^2)(L^*L)^\dagger K^*(a_{j,t} - F_{j,t}) \\ & + \sum_{i=1}^{p} (a_{j,t}, K\omega_i)\, \omega_i - (1/\alpha) \sum_{i=1}^{p} (l_{j,t}, (L^*)^\dagger K^* K\omega_i)\, \omega_i \end{aligned} \tag{7.22}$$

for $j = 0, 1,..., n-1$ and for $t \in [a, b]$.

For our numerical work in [7] it will be essential to establish the regularity of the function $f_{j,t}$ and to bound various of its derivatives. Both problems can be solved by utilizing the representation (7.22). Let us conclude this section by examining the regularity of $f_{j,t}$, leaving the derivative bounds to [7].

Consider each of the terms in (7.22). We know that $G_t^\dagger$ belongs to $H^{n-1}[a, b]$, so $G_{j,t}^\dagger$ is an element of $H^{n-j-1}[a, b]$, and hence, by the smoothing property of $L^\dagger$ on functions in $\mathscr{R}(L)$ we have

$$L^\dagger G_{j,t}^\dagger \in H^{2n-j-1}[a, b].$$

Also,

$$(L^*L)^\dagger K^*K\omega_i \in \mathscr{D}(L^*L) \subseteq H^{2n}[a, b],$$
$$(L^*L)^\dagger K^*(a_{j,t} - F_{j,t}) \in \mathscr{D}(L^*L) \subseteq H^{2n}[a, b],$$

and $\omega_i \in C^\infty[a, b]$. Thus, from (7.22) we obtain the following lemma on the global regularity of $f_{j,t}$:

LEMMA 7.6. *If* $f_{j,t} - T_x^{-1} g_{j,t} = T_x^{-1}(a_{j,t}, l_{j,t})$ *for* $j = 0, 1,..., n - 1$ *and for* $t \in [a, b]$, *then* $f_{j,t} \in H^{2n-j-1}[a, b]$.

Remark 7.7. From (7.8) it is clear that $G_{j,t}^\dagger \in C^\infty[a, t] \cap C^\infty[t, b]$. Also, since

$$L^\dagger G_{j,t}^\dagger(s) = \int_a^b G^\dagger(s, \xi)\, G_j^\dagger(t, \xi)\, d\xi, \qquad a \leqslant s \leqslant b, \tag{7.23}$$

by substituting the representations (7.7) and (7.8) into (7.23) it follows that

$$L^\dagger G_{j,t}^\dagger \in C^\infty[a, t] \cap C^\infty[t, b],$$

and each of the derivatives of $L^\dagger G_{j,t}^\dagger$ with respect to s exists for all $s \neq t$ and is bounded independent of t. In particular, from (7.22), Lemma 7.2, and (7.14)–(7.15) we have

$$\sup_{a \leqslant t \leqslant b} \| f_{j,t}^{(2n)} \| = \gamma_0 < \infty, \tag{7.24}$$

for $j = 0, 1,..., n - 1$, where the constant γ_0 depends on the linear operators K and L.

Remark 7.8. Let Δ: $a = t_0 < t_1 < \cdots < t_M = b$ a partition of $[a, b]$, and let t be one of the knots of Δ, say $t = t_k$. Then from the above we see that globally

$$f_{j,t_k} \in H^n[a, b],$$

while locally

$$f_{j,t_k} \in H^{2n}[t_{i-1}, t_i], \qquad i = 1,..., M,$$

for $j = 0, 1,..., n - 1$. This type of regularity will be exploited in [7].

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